Binary Opinion Dynamics with Biased Agents and Agents with Different Degrees of Stubbornness

Arpan Mukhopadhyay
Bell Labs - INRIA, Paris, France
Email: arpan.mukhopadhyay@inria.fr

Ravi R. Mazumdar
Electrical and Computer Engineering
University of Waterloo, Canada,
Email: mazum@uwaterloo.ca

Rahul Roy
Indian Statistical Institute, Delhi
Email: rahul@isid.ac.in

Abstract*—In this paper, we investigate the impact of random interactions between agents in a social network on the diffusion of opinions in the network. Opinion of each agent is assumed to be a binary variable and each agent is assumed to be able to interact with any other agent in the network. This models scenarios where every agent in the network has to choose from two available options and the size of the neighborhood of each agent is an increasing function of the total number agents in the network. It is assumed that each agent updates its opinion at random instants upon interacting with other randomly sampled agents. We consider two simple rules of interaction: (1) the *voter rule* in which the updating agent simply copies the opinion of another randomly sampled agent; (2) the *majority rule*, in which the updating agent samples multiple agents and adopts the majority opinion among the sampled agents and the agent itself. Under each rule, we consider two different scenarios which have not been considered in the literature thus far: (1) where the agents are 'biased' towards one of the opinions, (2) where different agents have different degrees of stubbornness. We show that the presence of biased agents reduces the consensus time for the voter rule exponentially as compared to the case where the agents are unbiased. For the majority rule model with biased agents, we show that the network reaches consensus on a particular opinion with high probability only when the initial fraction of agents having that opinion is above a certain threshold. For the majority rule model with stubborn agents, we observe *metastability* where the network switches back and forth between stable states spending long intervals in each state.

I. Introduction

With the widespread use of online social networking, opinions of individuals are constantly being shaped by social interactions. Understanding how individual opinions are affected by social interactions and what global opinion structure emerges from such interactions are important in many contexts such as economics, politics and psychology. Mathematical models of social interaction treat the opinion of each individual in a social network as a variable taking values in either a discrete or a continuous subset of the Euclidean space. Although this may seem too reductive to capture the complexity of choices made by real individuals, in everyday situations, individuals in a network are often faced with only a limited number of choices (often as few as two) concerning a specific issue, e.g., pro-/anti-government, Windows/Linux, Democrat/Republican, etc. Thus, a vast body of literature treats opinions of individuals as binary variables taking values in the set $\{0, 1\}$.

The interactions between agents in a social network are generally modeled using simple rules that capture the essential features of individuals in a society such as their tendency to mimic their neighbors or to conform with the majority opinion in local neighborhoods. One of the models, extensively analyzed in the literature, is the *voter model* [1]–[3] or the *voter rule*, where an agent randomly samples one of its neighbors at an instant when it decides to update its opinion. The updating agent then adopts the opinion of the sampled neighbor. This simple rule captures the tendency of an individual to mimic other individuals in the society. Because of its simplicity, the rule has been analyzed under a variety of network topologies [4] that assume connectedness of the underlying graph.[1] It is known that, under the voter rule, any connected network converges to a *consensus*, where all individuals adopt the same opinion. It is of interest to determine the probability with which consensus is reached on a specific opinion and the time it takes for the network to converge to the consensus state.

Another rule studied in this context is the *majority rule* model [5]–[7]. In it, instead of sampling a single individual, an updating agent consults multiple individuals while performing the update and adopts the choice of the majority of the sampled neighbors. This rule captures the tendency of the individuals to conform with the majority opinion in their local neighborhoods. In a fully connected network, the majority rule also leads to a consensus among agents. However, the rate at which consensus is reached is faster than that under the voter rule.

In all the prior works on the voter models and the majority rule models, it is assumed that an agent's decision to update its opinion does not depend on the current opinion of the agent. It is also assumed that all agents in the network have the same propensity to change their opinions. However, in a real scenario an agent may be 'biased' towards a specific opinion in the sense that if it holds its 'preferred' opinion, then the probability with which it updates its opinion is small. We may also encounter situations where some of the agents update their choices less frequently than others (irrespective of their current opinions). In this paper, we focus on these two scenarios.

[1]Connectedness implies that every individual is connected to every other individual either directly or via immediate neighbors.

978-0-9883045-1-2/16 $31.00 © 2016 ITC

A. Related literature

There is a rich and growing literature that studies diffusion of technologies and opinions in large social networks in both Bayesian and non-Bayesian settings. The voter models and the majority rule models fall under the non-Bayesian setting. One of the first models in the non-Bayesian setting was studied by DeGroot [8], where the agents were assumed to update their opinions (assumed to be continuous variables within a certain range) synchronously by averaging the opinions of their neighbors. This is equivalent to the synchronous average consensus algorithms considered in [9] and thus can be analyzed using similar techniques. The 'voter model' with binary opinions was first studied independently in [1] and [2]. It was assumed that an agent simply copies the opinion of a randomly sampled neighbor at an instant of update. Due to its simplicity, the voter model soon became popular and was analyzed under a variety of network topologies, e.g., finite integer lattices in different dimensions [3], [10], heterogeneous graphs [11], Erdos-Renyi random graphs and random geometric graphs [4] etc. In [12], [13], the voter model was studied under the presence of stubborn individuals who do not update their opinions. In such a scenario, the network cannot reach a consensus because of the presence of stubborn agents having both opinions. Using coalescing random walk techniques the average opinion in the network and the variance of opinions were computed at steady state. A model where the agents have continuous values of opinion in the interval $[0, 1]$ and the updates occur iteratively based on the minimization of a cost function that take into account an agent's past opinion was considered in [14].

The majority rule model was first introduced in [15], where it was assumed that, at every iteration, groups of random sizes are formed by the agents. Within each group, the majority opinion is adopted by all the agents. Under this rule, it was shown that consensus is achieved on a particular opinion with high probability only if the initial fraction of agents having that opinion is more than a certain critical value. Furthermore, the time to reach consensus was shown to scale as logarithm of the network size (number of agents). Similar models with fixed (odd) group size were considered in [5], [6]. It was shown that for finite dimensional integer lattices the consensus time grows as a power law in the number of agents in the network.

A deterministic version of the majority rule model, where an agent, instead of randomly sampling some of its neighbors, adopts the majority opinion among all its neighbors, is considered in [16]–[19]. In such models, given the graph structure of the network, the opinions of the agents at any time is a deterministic function of the initial opinions of the agents. The interest there is to find out the the initial distribution of opinions for which the network converges to some specific absorbing state. In social networks, where the neighborhood of each agent is large, such majority rule dynamics involves complex computation by each updating agent at each update instant. Our interest in this paper is on scenarios where the agents are mobile and do not have any fixed neighborhoods. We therefore consider a randomized version of the majority rule.

B. Contributions

In this paper, we study binary opinion dynamics under the voter model and the majority rule model. Under each model, we consider the following two scenarios: 1) where the agents are 'biased' towards a specific opinion. 2) where different agents have different propensities to change their opinions or different degrees of stubbornness. We make the following contributions

1) For the voter model with biased agents, we derive a closed form expression of the probability with which consensus is reached on the 'preferred' opinion. It is observed that this probability increases rapidly to 1 as the number of agents in the network grows. This is unlike the case with unbiased agents, where the probability to reach consensus on a particular opinion remains constant for all network sizes. Using mean field techniques, we derive an estimate of the average time taken for the network to reach consensus. It is observed that the mean consensus time grows as logarithm of the network size. This is in contrast to the case with unbiased agents, where the mean consensus time grows linearly with the number of agents.

2) For the voter model with differently stubborn agents, we show that the probability of reaching consensus on a particular state is independent of the network size. We also show that the time to reach consensus grows linearly with the total number of agents.

3) For the majority rule model with biased agents, we derive a closed form expression for the probability with which consensus is achieved on the preferred opinion. It is observed that, unlike the voter model, consensus is achieved on the preferred opinion (with high probability) only if the initial fraction of agents having that opinion is above a certain threshold. This threshold is determined from the mean field analysis of the model. An estimate of the mean consensus time is also found from the mean field model. It suggests that the mean consensus time grows as logarithm of the number of agents in the network.

4) Finally, we consider the majority rule model when there are 'stubborn' agents in the network. The stubborn agents are assumed to have fixed opinions at all times. Therefore, in this case consensus can never be reached. We analyze the equilibrium distribution of opinions among the non-stubborn agents using mean field techniques. Depending on the system parameters, the mean field is shown to have either multiple stable equilibrium points or a unique stable equilibrium point within the range of interest. As the system size grows, the equilibrium distribution of opinions among non-stubborn agents is shown to converge to a mixture of Dirac measures concentrated on the equilibrium points of the mean field. This suggests a *metastable* behavior of the system where the system moves back and forth between stable configurations, spending long intervals in each configuration. The conditions for metastability are obtained in terms of the system parameters.

The rest of the paper is organized as follows. In Section II, we introduce the voter model. In Subsections II-A and II-B, we analyze the voter model with 'biased' agents and agents having different degrees of stubbornness, respectively. Section III introduces the majority rule model. In Subsections III-A and III-B, we analyze the majority rule model with 'biased' and 'stubborn' agents, respectively. Finally, the paper is concluded in Section IV.

II. THE VOTER MODELS

Let us consider a network consisting of N social agents, where each agent can communicate with every other agent. The results derived in this paper also holds for cases where size of the neighborhood of each agent is $O(N)$. Opinion of each agent is assumed to be a binary variable taking values in the set $\{0, 1\}$. Initially, every agent adopts one of the two opinions. The agents then consider updating their opinions at points of independent unit rate Poisson processes associated with themselves. At a point of the Poisson process associated with itself, an agent either updates its opinion or retains its past opinion. In case the agent decides to update its opinion, it samples an agent uniformly at random (with replacement) from the network[2] and adopts the opinion of the sampled agent.

Below we consider two different scenarios: (1) where the agents are 'biased towards a specific opinion, and (2) where the agents have different propensities to change their past opinions.

A. The voter model with biased agents

We first consider the case where the agents are 'biased' towards one of the two opinions. Without loss of generality, we assume that all agents in the network prefer opinion $\{1\}$ to opinion $\{0\}$. This is modeled as follows: Each agent with opinion $i \in \{0, 1\}$ updates its opinion at a point of the unit rate Poisson process associated with itself with probability q_i and retains its opinion with probability $p_i = 1 - q_i$. This is equivalent to an agent with opinion i updating its opinion at all points of a Poisson process with rate q_i. In case the agent decides to update its opinion, the update occurs following the voter rule discussed in the beginning of this section. We assume $q_0 > q_1$ ($p_1 > p_0$) to imply that the agents having opinion $\{0\}$ update their opinions more frequently than the agents having opinion $\{1\}$. In the above sense, the agents are biased towards opinion $\{1\}$.

Clearly, in this case, the network gets absorbed (in a finite time) in a state where all the agents adopt the same opinion. This is referred to as the *consensus state*. Our interest is to find out the probability with which consensus is achieved on the preferred opinion $\{1\}$ starting from a state where a fixed proportion of agents are in state $\{1\}$. This probability is referred to as the *exit probability* of the network. We also intend to characterize the mean time to reach the consensus state.

[2]In the large N limit sampling with or without replacement does not make any difference.

The case $q_1 = q_0 = 1$ is referred to as the voter model with unbiased agents, which has been analyzed in [1], [2]. It is known that for unbiased agents the probability with which consensus is reached on a particular opinion starting from a state where α fraction of agents have that particular opinion is simply equal to α, which is independent of N. Furthermore, the expected time to reach consensus for large N is known to be $Nh(\alpha)$, where $h(\alpha) = -[\alpha \ln(\alpha) + (1 - \alpha) \ln(1 - \alpha)]$. We now proceed to characterize these quantities for the voter model with biased agents.

Let $X^{(N)}(t)$ denote the number of agents with opinion $\{1\}$ at time $t \geq 0$. Clearly, $X^{(N)}(\cdot)$ is a Markov process on state space $\{0, 1, \dots, N\}$, with possible jumps at the points of a rate N Poisson process. This rate N process is referred to as the *global clock*. All states, except the states 0 and N, form an open communicating class; the states 0 and N are the absorbing states. Therefore, with probability 1, the process gets absorbed in one of the absorbing states in a finite time.

Proposition *1:* The probability $E_N(n)$ with which the process $X^{(N)}(\cdot)$ gets absorbed in state N starting with state $n \in \{1, 2 \dots, N\}$ is given by $E_N(n) = \frac{1-r^n}{1-r^N}$, where $r = q_1/q_0 < 1$ and $E_N(0) = 0$.

Proof: Given that the process $X^{(N)}(\cdot)$ is in state k at one point of the global clock, it transits to state $k + 1$ at the next point of the global clock only if one of the agents having opinion $\{0\}$ updates its opinion to opinion $\{1\}$. The probability with which any one of the $N - k$ agents having opinion $\{0\}$ decides to update its opinion is given by $q_0 \times (N - k)/N$. The probability with which the updating agent samples an agent with opinion $\{1\}$ is given by k/N. Hence, the total probability with which the process $X^{(N)}(\cdot)$ transits from the state k to the state $k+1$ is given by $p(k \to k+1) = \frac{k(N-k)}{N^2} q_0$. Similarly, the probability of transition from the state k to the state $k - 1$ is given by $p(k \to k - 1) = \frac{k(N-k)}{N^2} q_1$. Therefore, the probability with which no transition occurs between two consecutive points of the global clock is $p(k \to k) = 1 - p(k \to k+1) - p(k \to k-1)$. Since $X^{(N)}(\cdot)$ is Markov, $E_N(n)$ must satisfy the following recursive relationship $E_N(n) = p(n \to n+1)E_N(n+1) + p(n \to n-1)E_N(n-1) + p(n \to n)E_N(n)$, which can be solved (using the transition rates given above) to yield the desired expression. ∎

In terms of the initial fraction $\alpha = n/N$ of agents having opinion $\{1\}$, the exit probability derived above can be expressed as $E_N(\alpha) = \frac{1-r^{N\alpha}}{1-r^N}$. Clearly, for $q_0 > q_1$, we have $r < 1$. Hence, as N increases the exit probability rapidly increases to 1 for all α. This is in contrast to the case with unbiased agents ($q_0 = q_1 = 1$) where the exit probability remains constant at α for all values of N.

We now characterize the mean time $\bar{T}_N(\alpha)$ to reach the consensus state starting from α fraction of agents having opinion $\{1\}$. To do so, we consider the empirical measure process $x^{(N)}(\cdot) = X^{(N)}(\cdot)/N$, which describes the evolution of the fraction of agents with opinion $\{1\}$. The process $x^{(N)}(\cdot)$ jumps from the state x to the state $x + 1/N$ when one of the $N(1 - x)$ agents having opinion $\{0\}$ updates (with probability q_0) its opinion by interacting with an agent with opinion $\{1\}$. Since the agents update their opinions at points of independent unit rate Poisson processes, the rate at which one of the $N(1-x)$ agents having opinion $\{0\}$ decides to update its opinion is $N(1-x)q_0$. The probability with which the updating agent interacts with an agent with opinion $\{1\}$ is x. Hence, the total rate of transition from x to $x+1/N$ is given by $r(x \to x+1/N) = q_0 N x(1-x)$. Similarly, the rate of transition from x to $x-1/N$ is given by $r(x \to x-1/N) = q_1 N x(1-x)$. From the above transition rates it can be easily seen that the generator of the process $x^{(N)}(\cdot)$ converges uniformly as $N \to \infty$ to the generator of the deterministic process $x(\cdot)$ which is the unique solution of the following differential equation

$$\dot{x}(t) = (q_0 - q_1)x(t)(1-x(t)). \tag{1}$$

Thus, by the classical results of Kurtz [20] we have that if $x^{(N)}(0) \Rightarrow x(0)$ as $N \to \infty$, then $x^{(N)}(\cdot) \Rightarrow x(\cdot)$ as $N \to \infty$, where $\Rightarrow$ denotes weak convergence. In other words, for large N, the process $x^{(N)}(\cdot)$ can be approximated by the deterministic process $x(\cdot)$ which is called the *mean field limit* of the system.

Since $q_0 > q_1$ and $x(t) \in [0,1]$ for all $t \geq 0$, we have from (1) that $\dot{x}(t) \geq 0$ for all $t \geq 0$. Hence, $x(t) \to 1$ as $t \to \infty$. The mean consensus time $\bar{T}_N(\alpha)$ for large N can therefore be approximated by the time taken by the process $x(t)$ to reach the state $1 - 1/N$ (which corresponds to the situation where all the agents except one agent have opinion $\{1\}$) starting with $x(0) = \alpha$. Hence, by solving (1), we obtain

$$\begin{aligned}\bar{T}_N(\alpha) &= \frac{1}{q_0 - q_1}\ln(N-1) - \frac{1}{q_0 - q_1}\ln\left(\frac{\alpha}{1-\alpha}\right) \\ &= O\left(\frac{1}{|q_0 - q_1|}\ln(N-1)\right)\end{aligned} \tag{2}$$

Clearly, the mean consensus time scales as $O(\ln N)$. This is in contrast to the voter model with unbiased agents where the mean consensus time is known to increase linearly with the network size N. Thus, in the case with biased agents, the network reaches the consensus state exponentially faster than that in the case with unbiased agents.

Numerical Results: In Figure 1, we plot the exit probability for both biased ($q_0 > q_1$) and unbiased ($q_0 = q_1 = 1$) cases as functions of the number of agents N for $\alpha = 0.2$. For the biased case, we have chosen $q_0 = 1, q_1 = 0.5$. We observe that in the biased case the exit probability rapidly increases to 1 with the increase N. This is in contrast to the unbiased case, where the exit probability remains constant at α for all N.

In Figure 2, we plot the mean consensus time $\bar{T}_N(\alpha)$ for both the biased and unbiased cases as functions of N for $\alpha = 0.4$. We observe that, in the biased case, the consensus state is reached in a time exponentially smaller than that in the unbiased case. This is because the bias of the agents towards one of the opinions drives the system to consensus much faster.

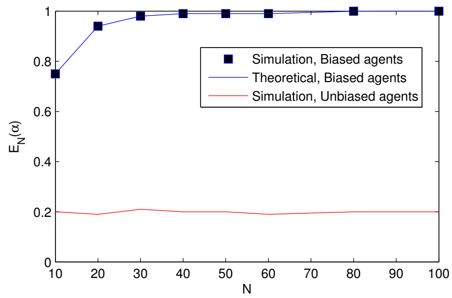

Fig. 1. Exit probability $E_N(\alpha)$ as a function of the number of agents N. Parameters: $q_0 = 1$, $q_1 = 0.5$, $\alpha = 0.2$.

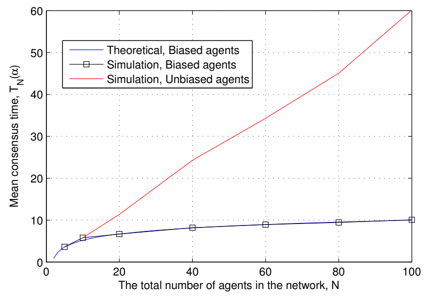

Fig. 2. Mean consensus time $\bar{T}_N(\alpha)$ as a function of the number of agents N. Parameters: $q_0 = 1$, $q_1 = 0.5$, $\alpha = 0.4$.

B. The voter model with agents having different degrees of stubbornness

We now consider the case where different agents have different propensities to change their opinions. We note that the case where some agents never update states was studied in [13]. To distinguish our work from this work we consider the following model: Each agent in the network is assumed to belong to one of the two disjoint sets $\mathcal{S}$ and $\mathcal{R}$. We denote by γ_S and $\gamma_R = 1 - \gamma_S$ the fractions of agents belonging to the sets $\mathcal{S}$ and $\mathcal{R}$, respectively. Each agent belonging to the set $\mathcal{S}$ ($\mathcal{R}$) updates its opinion with probability q_S (q_R) at a point of the unit rate Poisson process associated with itself and retains its opinion with probability $p_S = 1 - q_S$ ($p_R = 1 - q_R$). The updates occur according to the voter rule, discussed in the beginning of this section. The probabilities q_S and q_R determine the degrees of 'stubbornness' of agents belonging to the sets $\mathcal{S}$ and $\mathcal{R}$, respectively. We set $q_S < q_R$ to imply that the agents belonging to the set $\mathcal{S}$ update their opinions less frequently than the agents belonging to the set $\mathcal{R}$.

The evolution of the network can be described by a two dimensional Markov process $X^{(N)}(\cdot) = (X_S^{(N)}(\cdot), X_R^{(N)}(\cdot))$, where $X_S^{(N)}(t)$ and $X_R^{(N)}(t)$ denote the numbers of agents with opinion $\{1\}$ in sets $\mathcal{S}$ and $\mathcal{R}$, respectively, at time t. Let (m, n) be the state of the process at some instant. The process transits to state $(m+1, n)$ when one of the $N\gamma_S - m$ agents with opinion $\{0\}$ in the set $\mathcal{S}$ updates its opinion by interacting with an agent with opinion $\{1\}$. The rate at which any one of the $N\gamma_S - m$ agents having opinion $\{0\}$ in set $\mathcal{S}$ decides to update its opinion is $(N\gamma_S - m)q_S$. The probability with which the updating agent samples an agent with opinion $\{1\}$ from the entire network is $(m+n)/N$. Hence, the total rate of this transition is given by

$$r\left((m,n) \to (m+1,n)\right) = \frac{(N\gamma_S - m)(m+n)}{N} q_S \quad (3)$$

The rates of other possible transitions are similarly given by

$$r\left((m,n) \to (m,n+1)\right) = \frac{(N\gamma_R - n)(m+n)}{N} q_R \quad (4)$$

$$r\left((m,n) \to (m-1,n)\right) = \frac{m(N-m-n)}{N} q_S \quad (5)$$

$$r\left((m,n) \to (m,n-1)\right) = \frac{n(N-m-n)}{N} q_R \quad (6)$$

Proposition *2:* Let $E_N(\alpha_S, \alpha_R)$ denote the probability with which the network with N agents reaches a consensus state with all agents having opinion $\{1\}$ starting with α_S (resp. α_R) fraction of agents of the set $\mathcal{S}$ (resp. $\mathcal{R}$) having opinion $\{1\}$. Then

$$E_N(\alpha_S, \alpha_R) = \frac{q_R\gamma_S\alpha_S + q_S\gamma_R\alpha_R}{q_R\gamma_S + q_S\gamma_R}. \quad (7)$$

Proof: Let $\mathcal{F}_t = \sigma(X^{(N)}(s), 0 \le s \le t)$ denote the history of the process $X^{(N)}(\cdot)$ upto time $t \ge 0$. Consider the process $X_S^{(N)}(\cdot)/q_S + X_R^{(N)}(\cdot)/q_R$. Using the transition rates of the process $X^{(N)}(\cdot)$ it is easy to see that the conditional drift of the process from time t to time $t+h$ is given by

$$\begin{aligned}
&\mathbb{E}\left[\frac{X_S^{(N)}(t+h)}{q_S} + \frac{X_R^{(N)}(t+h)}{q_R} - \frac{X_S^{(N)}(t)}{q_S} - \frac{X_R^{(N)}(t)}{q_R} | \mathcal{F}_t\right] \\
&= \Bigg(\frac{r\left((m,n) \to (m+1,n)\right)}{q_S} \; \frac{r\left((m,n) \to (m,n+1)\right)}{q_R} \\
&- \frac{r\left((m,n) \to (m-1,n)\right)}{q_S} - \frac{r\left((m,n) \to (m,n-1)\right)}{q_R}\Bigg) h \\
&+ o(h) = o(h) \quad (8)
\end{aligned}$$

Thus, the process $X_S^{(N)}(\cdot)/q_S + X_R^{(N)}(\cdot)/q_R$ is an $\mathcal{F}_t$ martingale. Let T denote the random time the process $X^{(N)}(\cdot)$ hits the consensus state. Clearly, T is an $\mathcal{F}_t$ stopping time. Hence, using optional sampling theorem we have

$$\begin{aligned}
\mathbb{E}\left[\frac{X_S^{(N)}(T)}{q_S} + \frac{X_R^{(N)}(T)}{q_R}\right] &= \mathbb{E}\left[\frac{X_S^{(N)}(0)}{q_S} + \frac{X_R^{(N)}(0)}{q_R}\right] \\
&= \frac{N\gamma_S\alpha_S}{q_S} + \frac{N\gamma_R\alpha_R}{q_R} \quad (9)
\end{aligned}$$

The left hand side of the above equation can be written as

$$\begin{aligned}
\mathbb{E}\left[\frac{X_S^{(N)}(T)}{q_S} + \frac{X_R^{(N)}(T)}{q_R}\right] &= \left(\frac{N\gamma_S}{q_S} + \frac{N\gamma_R}{q_R}\right) E_N(\alpha_S, \alpha_R) \\
&+ 0 \times (1 - E_N(\alpha_S, \alpha_R)) \quad (10)
\end{aligned}$$

Hence we obtain

$$E_N(\alpha_S, \alpha_R) = \frac{\frac{N\gamma_S\alpha_S}{q_S} + \frac{N\gamma_R\alpha_R}{q_R}}{\frac{N\gamma_S}{q_S} + \frac{N\gamma_R}{q_R}} \quad (11)$$

which simplifies to (7). ■

Remark *1: From* (7) *we see that the exit probability does not depend on the number of agents N. We also observe that for $\alpha_S = \alpha_R = \alpha$, the exit probability is given by $E_N(\alpha, \alpha) = \alpha$, which is also independent of q_S and q_R.*

The mean time $\bar{T}_N(\alpha_S, \alpha_R)$ to reach consensus starting with α_S (resp. α_R) fraction of agents of the set $\mathcal{S}$ (resp. $\mathcal{R}$) having opinion $\{1\}$ can be computed using the first step analysis of the empirical measure process $x^{(N)}(\cdot) = (X_S^{(N)}(\cdot)/N\gamma_S, X_R^{(N)}(\cdot)/N\gamma_R)$. The process $x^{(N)}(\cdot)$ changes its state only at points of a rate N Poisson point process, referred to as the global clock. The probability $p((x,y) \to (x + 1/N\gamma_S, y))$ with which the process transits from the state (x,y) at one point of the global clock to the state $(x + 1/N\gamma_S, y)$ at the next point of the global clock is given by

$$p\left((x,y) \to \left(x + \frac{1}{N\gamma_S}, y\right)\right) = \gamma_S(1-x)(\gamma_S x + \gamma_R y)q_S. \quad (12)$$

Similarly, the probabilities for the other possible transitions are given by

$$p\left((x,y) \to \left(x, y + \frac{1}{N\gamma_R}\right)\right) = \gamma_R(1-y)(\gamma_S x + \gamma_R y)q_R \quad (13)$$

$$p\left((x,y) \to \left(x - \frac{1}{N\gamma_S}, y\right)\right) = \gamma_S x(1 - \gamma_S x - \gamma_R y)q_S \quad (14)$$

$$p\left((x,y) \to \left(x, y - \frac{1}{N\gamma_R}\right)\right) = \gamma_S y(1 - \gamma_S x - \gamma_R y)q_R \quad (15)$$

Since the process $x^{(N)}(\cdot)$ is Markov and the average gap between two points of the global clock is $1/N$, we have the following recursive relation

$$\bar{T}_N(x,y) = p\left((x,y) \to \left(x + \frac{1}{N\gamma_S}, y\right)\right) \times \left(\bar{T}_N\left(x + \frac{1}{N\gamma_S}, y\right) + \frac{1}{N}\right)$$
$$+ p\left((x,y) \to \left(x, y + \frac{1}{N\gamma_R}\right)\right)\left(\bar{T}_N\left(x, y + \frac{1}{N\gamma_R}\right) + \frac{1}{N}\right)$$
$$+ p\left((x,y) \to \left(x - \frac{1}{N\gamma_S}, y\right)\right)\left(\bar{T}_N\left(x - \frac{1}{N\gamma_S}, y\right) + \frac{1}{N}\right)$$
$$+ p\left((x,y) \to \left(x, y - \frac{1}{N\gamma_R}\right)\right)\left(\bar{T}_N\left(x, y - \frac{1}{N\gamma_R}\right) + \frac{1}{N}\right)$$
$$+ p\left((x,y) \to (x,y)\right)\left(\bar{T}_N(x,y) + \frac{1}{N}\right) \quad (16)$$

Now using (12), (13), (14), (15) and the Taylor series expansion of $\bar{T}_N(\cdot,\cdot)$ of second order around the point (x,y) we have that for large N

$$\gamma_R q_S (y-x)\frac{\partial \bar{T}(x,y)}{\partial x} + \frac{q_S((\gamma_S+1)x + \gamma_R y - 2x(\gamma_S x + \gamma_R y))}{2N\gamma_S}\frac{\partial^2 \bar{T}(x,y)}{\partial x^2} + \gamma_S q_P (x-y)\frac{\partial \bar{T}(x,y)}{\partial y} + \frac{q_R(\gamma_S x + (\gamma_R+1)y - 2y(\gamma_S x + \gamma_R y))}{2N\gamma_R}\frac{\partial^2 \bar{T}(x,y)}{\partial y^2} = -1 \quad (17)$$

with boundary condition $\bar{T}_N(0,0) = \bar{T}_N(1,1) = 0$. An approximate solution of the above partial differential equation is given as by

$$\bar{T}_N(x,y) = N\left(\frac{\gamma_S}{q_S} + \frac{\gamma_R}{q_R}\right) \times h\left(\frac{\gamma_S q_R}{\gamma_S q_R + \gamma_R q_S}x + \frac{\gamma_R q_S}{\gamma_S q_R + \gamma_R q_S}y\right), \quad (18)$$

where $h(z) = -(z\ln z + (1-z)\ln(1-z))$. The approximation is obtained by noting that the above solution is exact for the cases $\gamma_S = 1, \gamma_R = 0$ and $\gamma_S = 0, \gamma_R = 1$. Moreover, putting the solution in (17) we see that the terms containing first order partial derivatives vanish and the terms containing the second order partial derivatives simplify approximately to -1.

Numerical results: To numerically investigate how consensus time varies with the system size N, in Figure 3 we plot the mean consensus time of 1000 independent runs of a network with the following parameters: $q_S = 0.3$, $q_R = 1$, $\alpha_S = \alpha_R = 0.8$, $\gamma_S = \gamma_R = 0.5$. We observe that the mean consensus time grows linearly with N. In the figure, we have also plotted the mean consensus time obtained using (18). We observe a close match between the simulation result and the approximate result which suggests that the approximation provided in (18) is accurate.

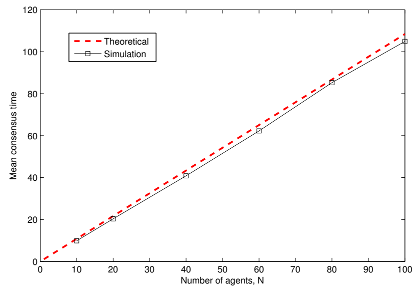

Fig. 3. Voter model with differently stubborn agents: Consensus time as a function of the network size N. Parameters: $q_S = 0.3$, $q_R = 1$, $\alpha_S = \alpha_R = 0.8$, $\gamma_S = \gamma_R = 0.5$.

III. The Majority Rule Models

In this section, we consider models where an agent, instead of interacting with a single agent, interacts with multiple agents at an update instant. As before, we assume that the agents in the network consider updating their opinions at points of independent, unit rate Poisson point processes. At a point of the Poisson process associated with itself, an agent either retains its opinion or updates it. If the agent decides to update its opinion, then it interacts with $2K$ ($K \geq 1$) agents sampled uniformly at random (with replacement) from the network and adopts the opinion held by the majority of the $2K+1$ agents which includes the $2K$ sampled agents and the updating agent itself. To simplify analysis, we focus on the $K = 1$ case, where each agent interacts with two agents at its update instant. The analysis is similar for $K > 1$ and simulations suggest that for large N varying K does not change the equilibrium properties of the network significantly.

As in the case of voter models, the decision of an agent to update its opinion is assumed to depend either i) on the current opinion held by the agent or ii) on the propensity of the agent to change opinions. Below we consider these two scenarios separately.

We note that in the majority rule model discussed above, only one agent updates its state, at each time step, by interacting with a group of randomly sampled neighbors. This is different from the previous models studied in the literature [5], [6], [15], where all members of a group of interacting agents were assumed to update their opinions simultaneously.

A. The majority rule model with biased agents

As in Section II-A, we first consider the case where the agents are 'biased' towards one of the two opinions. More specifically, we assume that an agent with opinion $i \in \{0,1\}$ updates its opinion with probability q_i at a point of the unit rate Poisson process associated with itself. The agent retains its opinion with probability $p_i = 1 - q_i$. In case the agent decides to update its opinion, the update occurs according to the majority rule discussed in the beginning of this section. We assume $q_0 > q_1$ to imply that agents with opinion $\{0\}$ update their opinions more frequently than agents with opinion $\{1\}$.

Proposition *3:* The probability $E_N(n)$ with which the process $X^{(N)}(\cdot)$ gets absorbed in state N starting from state $n \in \{1, 2, \ldots, N\}$ is given by $E_N(n) = \frac{1}{(1+r)^{N-1}} \sum_{k=0}^{n-1} \binom{N-1}{k} r^k$, where $r = q_1/q_0 < 1$ and $E_N(0) = 0$.

Proof: The proof is similar to the proof of Proposition 1 and uses the one-step analysis of the Markov chain $X^{(N)}(\cdot)$. We omit the proof due to space constraints. ■

We now proceed to characterize the mean consensus time of the network by analyzing the mean field limit of the empirical measure process $x^{(N)}(\cdot) = X^{(N)}(\cdot)/N$. As discussed for the voter model with biased agents it can be easily verified that in this case the mean field limit $x(\cdot)$ of the process $x^{(N)}(\cdot)$ is given by

$$\dot{x}(t) = (q_0 + q_1)x(t)(1 - x(t))\,(x(t) - \kappa_q)\,, \qquad (19)$$

where $\kappa_q = q_1/(q_0 + q_1)$.

From (19), it follows that the process $x(\cdot)$ has three equilibrium points at 0, 1, and κ_q, respectively. We now characterize the stability of these equilibrium points in the sense of the following definition:

Definition An equilibrium point $x_e \in [0, 1]$ of the process $x(\cdot)$ is called *stable* if there exists a non-empty set $S \subseteq [0, 1]$ containing x_e but $S \neq \{x_e\}$ such that for all $x(0) \in S$ we have $x(t) \to x_e$ as $t \to \infty$. If no such sets exist, then x_e is called an *unstable* equilibrium point. The equilibrium point x_e is said to be globally stable if for all $x(0) \in [0, 1]$ we have $x(t) \to x_e$ as $t \to \infty$.

If for some $t \geq 0$ we have $1 \geq x(t) > \kappa_q$, then (19) implies $\dot{x}(t) \geq 0$. Hence, for $x(0) \in (\kappa_q, 1]$ we have $x(t) \to 1$ as $t \to \infty$. Similarly, for $x(0) \in [0, \kappa_q)$ we have $x(t) \to 0$ as $t \to \infty$. Therefore, $0, 1$ are the stable equilibrium points of the process $x(\cdot)$, and κ_q is an unstable equilibrium point.

If $x^{(N)}(0) = \alpha = \kappa_q + \epsilon$ ($\epsilon > 0$), then, for large N, with high probability the process $x^{(N)}(\cdot)$ reaches the state 1 in a finite time. This is because for large N the path of the process $x^{(N)}(\cdot)$ is close to that of $x(\cdot)$ with high probability (by the mean field convergence result) and we have already shown that for $x(0) = \alpha > \kappa_q$, $x(t) \to 1$ as $t \to \infty$. Therefore, the mean consensus time for large N and $\alpha = \kappa_q + \epsilon$ can be approximated as the time taken by the process $x(\cdot)$ to reach the state $1 - 1/N$ from the state $\alpha = \kappa_q + \epsilon$. We denote the approximate mean consensus time by $\bar{T}_N(\kappa_q + \epsilon)$. Now, solving (19) with the above limits we obtain

$$\begin{aligned}\bar{T}_N\left(\kappa_q + \epsilon\right) = \frac{1}{q_0 + q_1}\Bigg[&\frac{1}{\kappa_q(1 - \kappa_q)}\ln(N(1 - \kappa_q) - 1) \\ &- \frac{1}{\kappa_q}\ln(N - 1) - \frac{1}{\kappa_q(1 - \kappa_q)}\ln\epsilon + \frac{1}{\kappa_q}\ln(\kappa_q + \epsilon) \\ &+ \frac{1}{1 - \kappa_q}\ln(1 - \kappa_q - \epsilon)\Bigg] \qquad (20)\end{aligned}$$

The expression for $\bar{T}_N(\alpha)$, for $\alpha = \kappa_q - \epsilon$, can be obtained similarly. From the above expressions, it is clear that the mean consensus time scales as $O(\ln N)$.

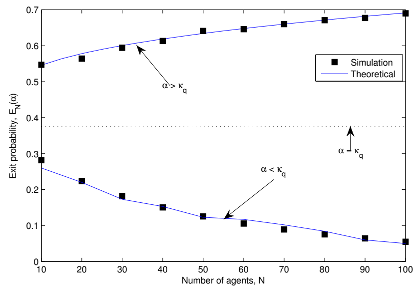

Fig. 4. Majority rule with biased agents: Exit probability $E_N(\alpha)$ as a function of the number of agents N. Parameters: $q_0 = 1$, $q_1 = 0.6$.

Numerical Results: In Figure 4, we plot the exit probability as a function of the total number N of agents in the network. The parameters are chosen to be $q_0 = 1$, $q_1 = 0.6$. We observe that for $\alpha > \kappa_q$, the exit probability increases to 1 with the increase in N and for $\alpha < \kappa_q$, the exit probability decreases to zero with the increase in N. This implies that consensus is achieved on the preferred opinion (opinion $\{1\}$) with high probability only if the initial fraction of agents having the preferred opinion is more than the threshold given by κ_q. This is unlike the voter model with biased agents where the consensus is achieved on the preferred opinion always with a higher probability.

B. The majority rule with stubborn agents

We now consider the majority rule model with agents having different propensities to change their opinions. Specifically, we assume that some of the agents in the network never update their opinions. We call these agents as the *stubborn* agents. The other agents, referred to as the *non-stubborn* agents, are assumed to update their opinions at all points of the Poisson processes associated with themselves. The updates occur according to the majority rule discussed in the beginning of this section. We denote by γ_i, $i \in \{0, 1\}$, the fraction of agents in network who are stubborn and have opinion i at all times. Thus, $(1 - \gamma_0 - \gamma_1)$ is fraction of non-stubborn agents in the network.

The presence of stubborn agents prevents the network from reaching a consensus state. This is because at all times there are at least $N\gamma_0$ stubborn agents having opinion $\{0\}$ and $N\gamma_1$ stubborn agents having opinion $\{1\}$. Furthermore, since each non-stubborn agent may interact with some stubborn agents at every update instant, it is always possible for the non-stubborn agent to change its opinion. Below we characterize the equilibrium fraction of non-stubborn agents having opinion $\{1\}$ in the network for large N using mean field techniques.

Let $x^{(N)}(t)$ denote the fraction of non-stubborn agents having opinion $\{1\}$ at time $t \geq 0$. Clearly, $x^{(N)}(\cdot)$ is a Markov process with possible jumps at the points of a rate $N(1 - \gamma_0 - \gamma_1)$ Poisson process. The process $x^{(N)}(\cdot)$ jumps from the state x to the state $x + 1/N(1 - \gamma_0 - \gamma_1)$ when one of the non-stubborn agents having opinion $\{0\}$ becomes active (which happens with rate $N(1-\gamma_0-\gamma_1)(1-x)$) and samples two agents with opinion $\{1\}$. The probability of sampling an agent having opinion $\{1\}$ from the entire network is $(1-\gamma_0-\gamma_1)x+\gamma_1$. Hence, the total rate at which the process transits from state x to the state $x+1/N(1-\gamma_0-\gamma_1)$ is given by $r\left(x \to x+\frac{1}{N(1-\gamma_0-\gamma_1)}\right) = N(1-\gamma_0-\gamma_1)(1-x)[(1-\gamma_0-\gamma_1)x+\gamma_1]^2$. Similarly, the rate of the other possible transition is given by $r\left(x \to x-\frac{1}{N(1-\gamma_0-\gamma_1)}\right) = N(1-\gamma_0-\gamma_1)x[(1-\gamma_0-\gamma_1)(1-x)+\gamma_0]^2$. Using the same line of arguments as discussed for the voter model with biased agents, it can be shown from above transition rates that the process $x^{(N)}(\cdot)$ converges weakly to the mean field limit $x(\cdot)$ which satisfies the following differential equation

$$\dot{x}(t) = (1-x(t))[(1-\gamma_0-\gamma_1)x(t)+\gamma_1]^2 - x(t) \times [(1-\gamma_0-\gamma_1)(1-x(t))+\gamma_0]^2. \quad (21)$$

We now study the equilibrium distribution π_N of the process $x^{(N)}(\cdot)$ for large N via the equilibrium points of the mean field $x(\cdot)$.

From (21) we see that $\dot{x}(t)$ is a cubic polynomial in $x(t)$. Hence, the process $x(\cdot)$ can have at most three equilibrium points in $[0,1]$. We first characterize the stability of these equilibrium points according to Definition III-A.

Proposition *4:* The process $x(\cdot)$ defined by (21) has at least one equilibrium point in $(0,1)$. Furthermore, the number of stable equilibrium points of $x(\cdot)$ in $(0,1)$ is either two or one. If there exists only one equilibrium point of $x(\cdot)$ in $(0,1)$, then the equilibrium point must be globally stable (attractive).

Proof: Define $f(x) = (1-x)[(1-\gamma_0-\gamma_1)x+\gamma_1]^2 - x[(1-\gamma_0-\gamma_1)(1-x)+\gamma_0]^2$. Clearly, $f(0) = \gamma_1^2 > 0$ and $f(1) = -\gamma_0^2 < 0$. Hence, there exists at least one root of $f(x) = 0$ in $(0,1)$. This proves the existence of an equilibrium point of $x(\cdot)$ in $(0,1)$.

Since $f(x)$ is a cubic polynomial and $f(0)f(1) < 0$, either all three roots of $f(x) = 0$ lie in $(0,1)$ or exactly one root of $f(x) = 0$ lies in $(0,1)$. Let the three (possibly complex and non-distinct) roots of $f(x) = 0$ be denoted by r_1, r_2, r_3, respectively. By expanding $f(x)$ we see that the coefficient of the cubic term is $-2(1-\gamma_0-\gamma_1)^2$. Hence, $f(x)$ can be written as

$$f(x) = -2(1-\gamma_0-\gamma_1)^2(x-r_1)(x-r_2)(x-r_3) \quad (22)$$

We first consider the case when $0 < r_1, r_2, r_3 < 1$ and not all of them are equal. Let us suppose, without loss of generality, that the roots are arranged in the increasing order, i.e., $0 < r_1 \le r_2 < r_3 < 1$ or $0 < r_1 < r_2 \le r_3 < 1$. From (22) and (21), it is clear that, if $x(t) > r_2$ and $x(t) > r_3$, then $\dot{x}(t) < 0$. Similarly, if $x(t) > r_2$ and $x(t) < r_3$, then $\dot{x}(t) > 0$. Hence, if $x(0) > r_2$ then $x(t) \to r_3$ as $t \to \infty$. Using similar arguments we have that for $x(0) < r_2$, $x(t) \to r_1$ as $t \to \infty$. Hence, r_1, r_3 are the stable equilibrium points of $x(\cdot)$. This proves that there exist at most two stable equilibrium points of the mean field $x(\cdot)$.

Now suppose that there exists only one equilibrium point of $x(\cdot)$ in $(0,1)$. This is possible either i) if there exists exactly one real root of $f(x) = 0$ in $(0,1)$, or ii) if all the roots of $f(x) = 0$ are equal and lie in $(0,1)$. Let r_1 be a root of $f(x) = 0$ in $(0,1)$. Now by expanding $f(x)$ from (22), we see that the product of the roots must be $\gamma_1^2/2(1-\gamma_0-\gamma_1)^2 > 0$. This implies that the other roots, r_2 and r_3, must satisfy one of the following conditions: 1) $r_2, r_3 > 1$, 2) $r_2, r_3 < 0$, 3) r_2, r_3 are complex conjugates, 4) $r_2 = r_3 = r_1$.

In all the above cases, we have that $(x-r_2)(x-r_3) \ge 0$ for all $x \in [0,1]$ with equality if and only if $x = r_1 = r_2 = r_3$. Hence, from (22) and (21), it is easy to see that $\dot{x}(t) > 0$ when $0 \le x(t) < r_1$ and $\dot{x}(t) < 0$ when $1 \ge x(t) > r_1$. This implies that $x(t) \to r_1$ for all $x(0) \in [0,1]$. In other words, r_1 is globally stable. ■

Hence, depending on the values of γ_0 and γ_1 there may exist of multiple stable equilibrium points of the mean field $x(\cdot)$. However, for every finite N, the process $x^{(N)}(\cdot)$ has a unique stationary distribution π_N (since it is irreducible on a finite state space). In the next result, we establish that any limit point of the sequence of stationary probability distributions $(\pi_N)_N$ is a convex combination of the Dirac measures concentrated on the equilibrium points of the mean field $x(\cdot)$ in $[0,1]$.

Theorem *1:* Any limit point of the sequence of probability measures $(\pi_N)_N$ is a convex combination of the Dirac measures concentrated on the equilibrium points of $x(\cdot)$ in $[0,1]$. In particular, if there exists a unique equilibrium point r of $x(\cdot)$ in $[0,1]$ then $\pi_N \Rightarrow \delta_r$, where δ_r denotes the Dirac measure concentrated at the point r.

Proof: We first note that since the sequence of probability measures $(\pi_N)_N$ is defined on the compact space $[0,1]$, it must be tight. Hence, Prokhorov's theorem implies that $(\pi_N)_N$ is relatively compact. Let π be any limit point of the sequence $(\pi_N)_N$. Then by the mean field convergence result we know that π must be an invariant distribution of the maps $\alpha \mapsto x(t,\alpha)$ for all $t \ge 0$, i.e., $\int \varphi(x(t,\alpha))d\pi(\alpha) = \int \varphi(\alpha)d\pi(\alpha)$, for all $t \ge 0$, and all continuous (and hence bounded) functions $\varphi : [0,1] \mapsto \mathbb{R}$. In the above, $x(t,\alpha)$ denotes the process $x(\cdot)$ started at $x(0) = \alpha$. Hence we have

$$\int \varphi(\alpha)d\pi(\alpha) = \lim_{t\to\infty} \int \varphi(x(t,\alpha))d\pi(\alpha) \quad (23)$$

$$= \int \varphi\left(\lim_{t\to\infty} x(t,\alpha)\right) d\pi(\alpha) \quad (24)$$

The second equality follows from the first by the Dominated convergence theorem and the continuity of φ. Now, let r_1, r_2, and r_3 denote the three equilibrium points of the mean field $x(\cdot)$. Hence, by Proposition 4 we have that for each $\alpha \in [0,1]$, $\varphi(\lim_{t\to\infty} x(t,\alpha)) = \varphi(r_1)I_{N_{r_1}}(\alpha) + \varphi(r_2)I_{N_{r_2}}(\alpha) + \varphi(r_3)I_{N_{r_3}}(\alpha)$, where for $i = 1,2,3$, $N_{r_i} \in [0,1]$ denotes the set for which if $x(0) \in N_{r_i}$ then $x(t) \to r_i$ as $t \to \infty$, and I denotes the indicator function. Hence, by (24) we have that for all continuous functions $\varphi : [0,1] \mapsto \mathbb{R}$

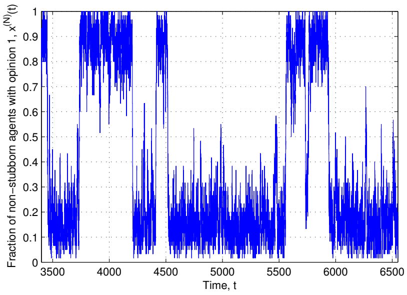

Fig. 5. Majority rule with stubborn agents: Sample path of the process $x^{(N)}(\cdot)$ with $N = 100$, $\gamma_0 = \gamma_1 = 0.2$.

$$\int \varphi(\alpha) d\pi(\alpha) = \varphi(r_1)\pi(N_{r_1}) + \varphi(r_2)\pi(N_{r_2}) + \varphi(r_3)\pi(N_{r_3}) \tag{25}$$

This proves that π must be of the form $\pi = c_1\delta_{r_1} + c_2\delta_{r_2} + c_3\delta_{r_3}$, where $c_1, c_2, c_3 \in [0, 1]$ are such that $c_1 + c_2 + c_3 = 1$. This completes the proof. ■

Thus, according to the above theorem, if there exists a unique equilibrium point of the process $x(\cdot)$ in [0,1], then the sequence of stationary distributions $(\pi_N)_N$ concentrates on that equilibrium point as $N \to \infty$. In other words, for large N, the fraction of non-stubborn agents having opinion $\{1\}$ (at equilibrium) will approximately be equal to the unique equilibrium point of the mean field.

If there exist multiple equilibrium points of the process $x(\cdot)$ then the convergence $x^{(N)}(\cdot) \Rightarrow x(\cdot)$ implies that at steady state the process $x^{(N)}(\cdot)$ spends intervals near the region corresponding to one of the stable equilibrium points of $x(\cdot)$. Then due to some rare events, it reaches, via the unstable equilibrium point, to a region corresponding to the other stable equilibrium point of $x(\cdot)$. This fluctuation repeats giving the process $x^{(N)}(\cdot)$ a unique stationary distribution. This behavior is formally known as *metastability*.

To demonstrate metastability, we simulate a network with $N = 100$ agents and $\gamma_0 = \gamma_1 = 0.2$. For the above parameters, the mean field $x(\cdot)$ has two stable equilibrium points at 0.127322 and 0.872678. In Figure 5, we show the sample path of the process $x^{(N)}(\cdot)$. We see that at steady state the process switches back and forth between regions corresponding to the stable equilibrium points of $x(\cdot)$. This provides numerical evidence of the metastable behavior of the finite system.

IV. Conclusion

In this paper, we analyzed the voter models the majority rule based model of social interaction under the presence of biased and differently stubborn agents. We observed that for the voter model, the presence of biased agents, reduces the mean consensus time exponentially in comparison to the voter model with unbiased agents. For the majority rule model with biased agents, we saw that the network reaches the consensus state with all agents adopting the preferred opinion only if the initial fraction of agents having the preferred opinion is more than a certain threshold value. Finally, we have seen that for the majority rule model with stubborn agents the network exhibits metastability, where it fluctuates between multiple stable configuration, spending long intervals in each configuration.

References

[1] R. A. Holley and T. M. Liggett, “Ergodic theorems for weakly interacting infinite systems and the voter model,” *Annals of Applied Probability*, vol. 3, no. 4, 1975.

[2] P. Clifford and A. Sudbury, “A model for spatial conflict,” *Biometrika*, vol. 60, no. 3, pp. 581–588, 1973.

[3] J. T. Cox, “Coalescing random walks and voter model consensus times on the torus in $\mathbb{Z}^d$,” *Annals of Probability*, vol. 17, no. 4, pp. 1333–1366, 1989.

[4] M. E. Yildiz, R. Pagliary, A. Ozdaglar, and A. Scaglione, “Voting models in random networks,” in *Information Theory and Applications Workshop (ITA)*, pp. 1–7, 2010.

[5] P. L. Krapivsky and S. Redner, “Dynamics of majority rule in two-state interacting spin systems,” *Physical Review Letters*, vol. 90, no. 23, p. 238701, 2003.

[6] P. Chen and S. Redner, “Consensus formation in multi-state majority and plurality models,” *Journal of Physics A: Mathematical and General*, vol. 38, no. 33, p. 7239, 2005.

[7] P. Chen and S. Redner, “Majority rule dynamics in finite dimensions,” *Physical Review E*, vol. 71, no. 3, p. 036101, 2005.

[8] M. H. DeGroot, “Reaching a consensus,” *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 118–121, 1974.

[9] J. N. Tsitsiklis, *Problems in decentralized decision making and computation*. PhD thesis, M.I.T, 1984.

[10] P. L. Krapivsky, “Kinetics of monomer-monomer surface katalytic reactions,” *Physical Review A*, vol. 45, no. 2, 1992.

[11] V. Sood and S. Redner, “Voter model on heterogeneous graphs,” *Physical Review Letters*, vol. 94, no. 17, 2005.

[12] M. Mobilia, A. Petersen, and S. Redner, “On the role of zealotry in the voter model,” *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2007, no. 8, p. P08029, 2007.

[13] E. Yildiz, A. Ozdaglar, D. Acemoglu, A. Saberi, and A. Scaglione, “Binary opinion dynamics with stubborn agents,” *ACM Transactions on Economics and Computation*, vol. 1, no. 4, pp. 1–30, 2013.

[14] J. Ghaderi and R. Srikant, “Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate,” *Autimatica*, vol. 50, no. 12, pp. 3209 – 3215, 2014.

[15] S. Galam, “Minority opinion spreading in random geometry,” *European Journal of Physics*, vol. 25, no. 4, pp. 403–406, 2002.

[16] E. Mossel, J. Neeman, and O. Tamuz, “Majority dynamics and aggregation of information in social networks,” *Autonomous Agents and Multi-Agent Systems*, vol. 28, no. 3, pp. 408–429, 2014.

[17] P. Flocchini, E. Lodi, F. Luccio, L. Pagli, and N. Santoro, “Dynamic monopolies in tori,” *Discrete Applied Mathematics*, vol. 137, no. 2, pp. 197–212, 2004.

[18] G. Moran, “The r-majority vote action on 0-1 sequences,” *Discrete Mathematics*, vol. 132, no. 1, pp. 145 – 174, 1994.

[19] Z. Agur, A. Fraenkel, and S. Klein, “The number of fixed points of the majority rule,” *Discrete Mathematics*, vol. 70, no. 3, pp. 295 – 302, 1988.

[20] T. G. Kurtz, “Solutions of ordinary differential equations as limits of pure jump markov processes,” *Journal of Applied Probability*, vol. 7, no. 1, pp. 49–58, 1970.